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【原著論文】

n 階行列式的降階法
The Reductive Algorithm for Determinant of a Matrix

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Abstract

Determinants are very efficient computational method to solve system of linear equations（Cramer’s rule）, the determinant of 2-by-2 matrix is the product of entries on the main diagonal minus the product of the other entries .Thus 2-by-2 determinants can usually be computed mentally. In principle, any determinant can be calculated from the formula of expansion, but this involves formidable amounts of arithmetic if the dimension is at all large. We need a new method of calculation and prove it, which is similar in 2-by-2 cases and involves less arithmetic than the method of Laplace expansion.

Keywords：determinant, matrix

摘要

我們在解線性方程組時，行列式解法是一種有效而且直接的方法（克萊瑪法則）。而二階行列式的計算僅須求兩對角線乘積之差即可，甚至可以利用心算求值，可是到了高階，雖然可以用公式展開，可是計算式子會非常的繁雜，所以我們找尋一種方法並證明它能夠擁有二階好計算的優點，又能避免高階展開時的繁雜龐大算式。

關鍵詞：行列式，矩陣

Introduction

The definition of determinant is a numerical-valued function defined on the set of all square matrices. The value of this function for a matrix A is called the determinant of A and is written det(A). Our definition of determinant is inductive , First, we define 1-by-1 matrices, and then for each n define the determinant of a n-by-n matrix in term of determinant of certain (n-1)-by-(n-1) matrix.

Definition: For a 1-by-1 matrix $A=[a]$，we define $\det A = a$

Definition:

For any n-by-n matrix A, Let i , j be a pair of integers between 1 and n. the matrix obtained by deleting the ith row and jth column of A, we obtain an (n-1)-by-(n-1) matrix, which is called the ijth minor of A and is denoted by A_{ij} . It looks like this:

$$A_{ij} = \begin{bmatrix} a_{11} & & \vdots & & a_{1n} \\ & & \vdots & & \\ \cdots & \cdots & a_{ij} & \cdots & \cdots \\ & & \vdots & & \\ a_{n1} & & \vdots & & a_{nn} \end{bmatrix}$$

The general case of n-by-n determinant is done by induction. We give an expression for the determinant of an n-by-n matrix in terms of determinants of (n-1)-by-(n-1) matrices. Let i, j be integers, $1 \le i,\ j \le n$. We define

(1) Expansion of the determinant according to the ith row :

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

(2) Expansion of the determinant according to the jth column:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

Example：For a 2-by-2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

By definition of the minor of A，

$A_{11} = d$, $A_{12} = c$,

we expand A by the first row,

$\det(A) = aA_{11} - bA_{12}$, $\det(A) = ad - bc$

The determinant of a 2-by-2 matrix is the product of the entries on the main diagonal minus the product of the other entries.

Properties of the determinant

To compute determinants efficiently, we need the following properties. Let A , B , C be 3 n-by-n matrices

Property 1. If B is obtained from A by multiplying some column(row) by a number k, then

$\det(B) = k \det(A)$

Property 2. If A , B , C are identical except in the jth column, and suppose that the jth column of C is the sum of the jth columns of A and B ,Then

$\det(A) = \det(B) + \det(C)$

Property 3. If B is obtained from A by exchanging two adjacent column(row), then

$\det(A) = \det(B)$

Property 4. If B is obtained from A by exchanging any two column(row), then

$\det(A) = -\det(B)$

Property 5. If any two column of A are identical, then $\det(A) = 0$

Property 6. If B is obtained from A by adding a numerical multiple of one column to another, then

$\det(A) = \det(B)$

Main Result

Theorem. (The method of reduction)

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \vdots \\ a_{n1} & & & a_{nn} \end{bmatrix}$

be an n-by-n matrix

If $a_{11} \neq 0$

Then $\det(A) = \dfrac{1}{a_{11}^{n-2}} \det(A^*)$

Where

$$A^* = \begin{bmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \vdots & & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{bmatrix}$$

Proof

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \vdots \\ a_{n1} & & & a_{nn} \end{vmatrix}$$

By property 6,we plus $\left(-\frac{a_{i1}}{a_{11}}\right)$ multiplies the first row to the ith row， $2 \le i \le n$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} - \frac{a_{12}a_{21}}{a_{11}} & & a_{2n} - \frac{a_{1n}a_{21}}{a_{11}} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} - \frac{a_{12}a_{n1}}{a_{11}} & & a_{nn} - \frac{a_{1n}a_{n1}}{a_{11}} \end{vmatrix}$$

By property 1, we multiply the ith row by a_{11} for $2 \le i \le n$.

Our determinant is then equal to

$$\frac{1}{a_{11}^{n-1}} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & b_{22} & & b_{2n} \\ \vdots & & & \vdots \\ 0 & b_{n2} & & b_{nn} \end{bmatrix}$$

where $b_{ij} = a_{11}a_{ij} - a_{1j}a_{i1}, 2 \le i, j \le n$

According to the expansion by first column, we get

$$\det(A) = \frac{a_{11}}{a_{11}^{n-1}} \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$$

$$= \frac{1}{a_{11}^{n-2}} \begin{bmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \vdots & & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{bmatrix}$$

This proves what we want and the previous formula of reduction is similar in 2-by-2 cases which is more convenient for calculation

Example1. Compute the determinant

$$\begin{vmatrix} 2 & -1 & 0 & 3 \\ -1 & 2 & 1 & 0 \\ 1 & 0 & 2 & -4 \\ 2 & 3 & 1 & 1 \end{vmatrix}$$

1. By formula of Laplace expansion:
We expand the determinant according to the first row

$$\begin{vmatrix} 2 & -1 & 0 & 3 \\ -1 & 2 & 1 & 0 \\ 1 & 0 & 2 & -4 \\ 2 & 3 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & -4 \\ 3 & 1 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 1 & 0 \\ 1 & 2 & -4 \\ 2 & 1 & 1 \end{vmatrix}$$

$$-3\begin{vmatrix} -1 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

=2 (4 –12+ 0 –0– (–8)) +(–2–8+ 0 – 0 – 1– 4) –3(0 + 8+3 –0 –2 –(–6))

=2(0)+(–15)– 3 (15)

$= -60$

2. By reductive Algorithm:

For the convenience of computation $(a_{11} = 1)$; by property 6, we add the first row from the second row.

$$\begin{vmatrix} 2 & -1 & 0 & 3 \\ -1 & 2 & 1 & 0 \\ 1 & 0 & 2 & -4 \\ 2 & 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 3 \\ -1 & 2 & 1 & 0 \\ 1 & 0 & 2 & -4 \\ 2 & 3 & 1 & 1 \end{vmatrix}$$

By theorem of reduction, our determinant is then equal to

$$\begin{vmatrix} 3 & 2 & 3 \\ -1 & 1 & -7 \\ 1 & -1 & -5 \end{vmatrix}$$

By property 3, the first and the third rows are interchanged, the determinant changes by a sign. Our determinant is equal to

$$-\begin{vmatrix} 1 & -1 & -5 \\ -1 & 1 & -7 \\ 3 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -5 \\ 1 & -1 & 7 \\ 3 & 2 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 12 \\ 5 & 18 \end{vmatrix} = -60$$

Example2. (the Vandermond determinant)

Evaluate the determinant

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$$

By our reductive algorithm, we derives

$$\begin{vmatrix} b-a & b^2-a^2 & b^3-a^3 \\ c-a & c^2-a^2 & c^3-a^3 \\ d-a & d^2-a^2 & d^3-a^3 \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)\begin{vmatrix} 1 & b+a & b^2+ba+a^2 \\ 1 & c+a & c^2+ca+a^2 \\ 1 & d+a & d^2+da+a^2 \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)\begin{vmatrix} c-b & c^2+ca-b^2-ba \\ d-b & d^2+da-b^2-ba \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)\begin{vmatrix} c-b & (c-b)(c+b)+a(c-b) \\ d-b & (d-b)(d+b)+a(d-b) \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)\begin{vmatrix} c-b & (c-b)(c+b+a) \\ d-b & (d-b)(d+b+a) \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)\begin{vmatrix} 1 & (c+b+a) \\ 1 & (d+b+a) \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$$

Numerical Experiment

we generate a *n*-by-*n* random matrix and calculate the corresponding determinant by various methods, for instance, for $n = 7$, a matrix is randomly chosen as follows:

RandomMatrix:=

$$\begin{bmatrix} 93 & -90 & 97 & -11 & -73 & 11 & 92 \\ -32 & 31 & 23 & -96 & -62 & 72 & -83 \\ -76 & 77 & 53 & -5 & -96 & 56 & -30 \\ 47 & -70 & -63 & -7 & 41 & 34 & 98 \\ 97 & 19 & 71 & 82 & 37 & -97 & -85 \\ 1 & -15 & 100 & 91 & -40 & 12 & -63 \\ 6 & -80 & 45 & 78 & -88 & 22 & -25 \end{bmatrix}$$

We make a CPU time comparison between

Gaussian Elimination, Laplace Expansion and our Reductive Algorithm, the result are 0.30e-1, .31e-1, 0.20e-1(see Fig1) which are unapparent in this case.

method	determinant	CPU Time/sec
Gaussian Elimination	-185839574069550	0.30e-1
Laplace Expansion	-185839574069550	0.31e-1
Reductive Algorithm	-185839574069550	0.20e-1

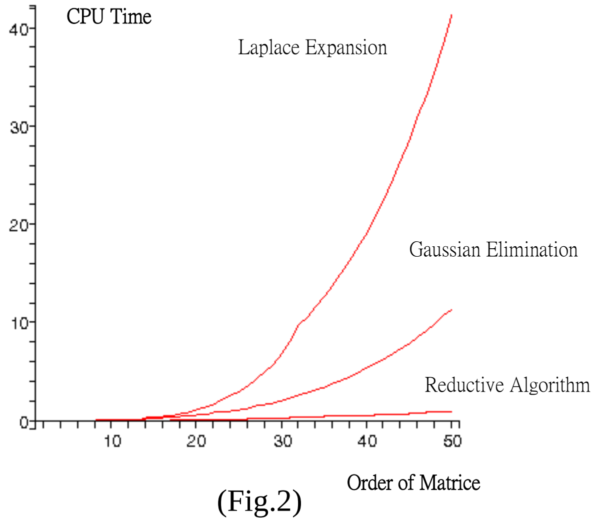
(Fig.1)

Maple 9 Program

In order to test how efficient about our reductive algorithm as *n* large, we wrote a computer program to calculate CPU time(in the programming language Maple 9). This program test determinants of the order in ranges from 2 to 50 in step of 1, all the components of the random matrix are integers from -100 to 100. The calculation took about 10 minutes on a 1667MHz AMD Athlon XP 2000+.

```
>with(LinearAlgebra): #access the Linear Algerbra package
 f:=proc(x)
local a,m,B,Gaussiantime,st;
a:=x:m:=a:
B:=RandomMatrix(m,m,generator
=-100..100); #create a random matrix
st:=time():
Determinant(B,method=algnum);
Gaussiantime:=time()-st;
end:
f1:=proc(x)
local
a,m,B,Expansion,Expansiontime
,st,s,k;
a:=x:m:=a:
B:=RandomMatrix(m,m,generator
=-100..100);
st:=time():
s:=0:
for  k from 1 to m do
Expansion:=Minor(B,k,1,
output=['matrix']):
s:=s+(-1)^(k+1)*B[k,1]*Determ
inant(Expansion):
end do:
Expansiontime:=time()-st:
end:
f2:=proc(x)
local
a,m,B,Reductive,Redutime,st,f
;
a:=x:m:=a:
B:=RandomMatrix(m,m,generator
=-100..100);
st:=time():
f:=(i,j)->
(B[1,1]*B[i+1,j+1]-B[1,j+1]*B
[i+1,1]):
Reductive:=Matrix(m-1,f);
Determinant(Reductive)/(B[1,1
]^(m-2));
Redutime:= time()-st;
end:
>
data2:=seq([i,time(f(i))],i=2
..50):
>
data3:=seq([i,time(f1(i))],i=
2..50):
>
data4:=seq([i,time(f2(i))],i=
2..50):
> with(plots):#access the plots package
g1:=plot([data2],thickness=3,
style=point):
g2:=plot([data3],thickness=2)
:
g3:=plot([data4],thickness=3)
:
display(g1,g2,g3);
```

Program Output



(Fig.2)

The resulting plot (see Fig.2) show us the elapsed time of 3 methods of computing determinants as n get larger, it supplies numerical evidence that the efficiency of the reductive algorithm is the best.

Conclusion

Determinants play a key role in the system of linear equations, there is no disputing the beauty facts of Cramer's rule. In general, any n-by-n determinant can be done by induction but it is a messy job to expand the determinant of a matrix–after all there are n! terms in the expansion. This paper provides an efficient algorithm which will facilitate such work.

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